

Integer Powers of Complex Tridiagonal and Anti-Tridiagonal Matrices

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Abstract

In this paper, we derive the general expression of the r -th power for some n -square complex tridiagonal matrices. Also one type is given eigenvalues and eigenvectors of complex anti-tridiagonal matrices. Additionally, we obtain the complex factorizations of Fibonacci polynomials.

1 Introduction

Elouafi and Hadj [1] offered tridiagonal matrix powers and inverse. Gutiérrez [2,3] obtained a general expression for the entries of the q -th power ($q \in \mathbb{N}$) of the $n \times n$ complex tridiagonal matrix $\text{tridiag}_n(a_1, a_0, a_{-1})$ for all $n \in \mathbb{N}$ and $2(q-1) \leq n$. Rimas [4-8] enquired the arbitrary positive integer powers for some tridiagonal matrices. Öteleş and Akbulak [9,10] generalized Rimas's the some results and get complex factorization formula for the generalized Fibonacci-Pell numbers.

Let

$$A := \begin{bmatrix} a & 2b & & & & 0 \\ b & a & b & & & \\ & b & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & b & a & 2b \\ 0 & & & & b & a \end{bmatrix} \quad (1)$$

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and

$$A^\dagger := \begin{bmatrix} a & b & & & 0 \\ b & a & -b & & \\ & -b & a & b & \\ & & b & a & -b \\ & & & -b & a & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix} \quad (2)$$

where $b \neq 0$ and $a, b \in \mathbb{C}$. In this paper, we want to r -th power obtain of an n -square complex tridiagonal matrices in (1) and (2).

We derive expression of the r -th power ($r \in \mathbb{N}$) a matrix applying the well-known expression $G^r = SJ^rS^{-1}$ [13], where J is the Jordan's form of the matrix G and S is the transforming matrix of G . We need the eigenvalues and eigenvectors of the matrices A and A^\dagger , respectively, to calculate transforming matrices.

Let Q be the following $n \times n$ tridiagonal matrix

$$Q := \begin{bmatrix} 0 & 2 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 0 & 2 \\ & & & & 1 & 0 \end{bmatrix}. \quad (3)$$

Then, the eigenvalues of Q is

$$\mu_k = 2 \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad k = 1, 2, \dots, n \quad [4].$$

The Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$ are defined as [14]

$$T_n(x) = \cos(n \arccos x), \quad -1 \leq x \leq 1 \quad (4)$$

and

$$U_n(x) = \frac{\sin((n+1) \arccos x)}{\sin(\arccos x)}, \quad -1 \leq x \leq 1. \quad (5)$$

All roots of the polynomial $U_n(x)$ are included in the interval $[-1, 1]$ and can be found using the relation

$$x_{nk} = \cos \left(\frac{k\pi}{n+1} \right), \quad k = 1, 2, \dots, n. \quad (6)$$

2 Eigenvalues and eigenvectors of A and A^\dagger

Theorem 1 *Let A be as in (1). Then the eigenvalues and eigenvectors of the matrix A are*

$$\lambda_k = a + 2b \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad k = 1, 2, \dots, n \quad (7)$$

and

$$x_{kj} = \begin{cases} T_{k-1} \left(\frac{\delta_j}{2} \right); & k = 1, 2, \dots, n-1 \\ \frac{1}{2} T_{k-1} \left(\frac{\delta_j}{2} \right); & k = n \end{cases} \quad j = 1, 2, \dots, n \quad (8)$$

where $\delta_j = \frac{\lambda_j - a}{b}$ and $T_k(x)$ is Chebyshev polynomial of the first kind.

Proof. Let B be as in the following $n \times n$ tridiagonal matrix:

$$B := \begin{bmatrix} \frac{a}{b} & 2 & & & & \\ 1 & \frac{a}{b} & 1 & & & \\ & 1 & \frac{a}{b} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \frac{a}{b} & 2 \\ & & & & 1 & \frac{a}{b} \end{bmatrix} \quad (9)$$

Then, the characteristic polynomials of B are

$$D_n(\alpha) = (\alpha^2 - 4)P_{n-2}(\alpha) \quad (10)$$

where $\alpha = \lambda - \frac{a}{b}$ and

$$P_n(\alpha) = \alpha P_{n-1}(\alpha) - P_{n-2}(\alpha) \quad (11)$$

with initial conditions $P_0(\alpha) = 1$, $P_1(\alpha) = \alpha$, $P_2(\alpha) = \alpha^2 - 1$.

Solution of difference equation in (11) is $P_n(\alpha) = U_n(\frac{\alpha}{2})$ here $U_n(x)$ is Chebyshev polynomial of the second kind. So the equality in (10) written as

$$D_n(\alpha) = (\alpha^2 - 4)U_{n-2}(\frac{\alpha}{2}).$$

Then, we have

$$\alpha_k = 2 \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad k = 1, 2, \dots, n.$$

So, the eigenvalues of A are

$$\lambda_k = a + 2b \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad \text{for } k = 1, 2, \dots, n.$$

Components eigenvectors of the matrix A are the solutions of the following homogeneous linear equations system:

$$(\lambda_k I_n - A)x = 0 \quad (12)$$

where λ_k is the k -th eigenvalue of the matrix A ($k = 1, 2, \dots, n$). The following equations system (12) written, we possess

$$\left. \begin{aligned} (\lambda_k - a)x_1 - 2bx_2 &= 0 \\ -bx_1 + (\lambda_k - a)x_2 - bx_3 &= 0 \\ -bx_2 + (\lambda_k - a)x_3 - bx_4 &= 0 \\ &\vdots \\ -bx_{n-2} + (\lambda_k - a)x_{n-1} - 2bx_n &= 0 \\ -bx_{n-1} + (\lambda_k - a)x_n &= 0 \end{aligned} \right\} \quad (13)$$

Dividing all terms of the each equation in system (13) by $b \neq 0$, substituting $\delta_j = \frac{\lambda_j - a}{b}$ ($j = 1, 2, \dots, n$). Since rank of the system is $n - 1$; choosing $x_1 = 1$ and solving the set of the system (13) as regards x_1 ,

$$x_{kj} = \begin{cases} T_{k-1}\left(\frac{\delta_j}{2}\right); & k = 1, 2, \dots, n-1 \\ \frac{1}{2}T_{k-1}\left(\frac{\delta_j}{2}\right); & k = n \end{cases} \quad j = 1, 2, \dots, n$$

where $\delta_j = \frac{\lambda_j - a}{b}$ and $T_k(x)$ is Chebyshev polynomial of the first kind. ■

Theorem 2 Let A^\dagger be as in (2). Then the eigenvalues and eigenvectors of the matrix A^\dagger are

$$\lambda_k^\dagger = a - 2b \cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, 2, \dots, n \quad (14)$$

and

$$y_{kj} = r_{k-1} U_{k-1}\left(\frac{\psi_j}{2}\right); \quad j, k = 1, 2, \dots, n \quad (15)$$

here $\psi_j = \frac{\lambda_j^\dagger - a}{b}$, $r_{k-1} = \begin{cases} 1, & k-1 \equiv 0 \text{ or } 1 \pmod{4} \\ -1, & k-1 \equiv 2 \text{ or } 3 \pmod{4} \end{cases}$ and $U_k(x)$ is Chebyshev polynomial of the second kind.

Proof. Let

$$B^\dagger := \begin{bmatrix} \frac{a}{b} & 1 & & & 0 \\ 1 & \frac{a}{b} & -1 & & \\ & -1 & \frac{a}{b} & 1 & \\ & & 1 & \frac{a}{b} & -1 \\ & & & -1 & \frac{a}{b} & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}. \quad (16)$$

Let Q^\dagger be the following $n \times n$ tridiagonal matrix

$$Q^\dagger := \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & -1 & & \\ & -1 & 0 & 1 & \\ & & 1 & 0 & -1 \\ & & & -1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}. \quad (17)$$

The eigenvalues of Q^\dagger are defined by the roots of the characteristic equation

$$|Q^\dagger - \theta I| = 0.$$

Let

$$D_n^\dagger(\theta) = \begin{vmatrix} \theta & 1 & & & \\ 1 & \theta & -1 & & \\ & -1 & \theta & 1 & \\ & & 1 & \theta & -1 \\ & & & -1 & \theta & \ddots \\ & & & & \ddots & \ddots \end{vmatrix} \quad (18)$$

and

$$P_n(\theta) = \begin{vmatrix} \theta & 1 & & & \\ 1 & \theta & 1 & & \\ & 1 & \theta & 1 & \\ & & 1 & \theta & 1 \\ & & & 1 & \theta & \ddots \\ & & & & \ddots & \ddots \end{vmatrix} \quad (19)$$

here $\theta \in \mathbb{R}$. Then

$$|Q^\dagger - \theta I| = D_n^\dagger(\theta)$$

and

$$D_n^\dagger(\theta) = P_n(\theta). \quad (20)$$

Let us prove by (20) the inductive method. For the basis step, we possess

$$\begin{aligned} D_1^\dagger(\theta) &= \theta = P_1(\theta) \\ D_2^\dagger(\theta) &= \theta^2 - 1 = P_2(\theta) \\ D_3^\dagger(\theta) &= \theta^3 - 2\theta = P_3(\theta). \end{aligned}$$

We suppose $D_{n-1}^\dagger(\theta) = \varkappa = P_{n-1}(\theta)$ and $D_n^\dagger(\theta) = \varrho = P_n(\theta)$ for $n \geq 3$. The well known recurrence relations is

$$|H(n)| = h_{n,n} |H(n-1)| - h_{n-1,n} h_{n,n-1} |H(n-2)| \quad [15]. \quad (21)$$

From (21), if n is a positive odd integer, since $h_{n,n} = \theta$, $h_{n-1,n} = -1$, $h_{n,n-1} = -1$, $D_{n-1}^\dagger(\theta) = \varkappa$ and $D_n^\dagger(\theta) = \varrho$,

$$D_{n+1}^\dagger(\theta) = \theta\varrho - (-1)(-1)\varkappa = \theta\varrho - \varkappa,$$

if n is a positive even integer, since $h_{n,n} = \theta$, $h_{n-1,n} = 1$ and $h_{n,n-1} = 1$,

$$D_{n+1}^\dagger(\theta) = \theta\varrho - \varkappa$$

and since $h_{n,n} = \theta$, $h_{n-1,n} = 1$, $h_{n,n-1} = 1$, $P_{n-1}(\theta) = \varkappa$ and $P_n(\theta) = \varrho$,

$$P_{n+1}(\theta) = \theta\varrho - \varkappa.$$

For $\forall n \in \mathbb{Z}$, we have $D_n^\dagger(\theta) = P_n(\theta)$. From (11), we obtain

$$D_n^\dagger(\theta) = U_n \left(\frac{\theta}{2} \right).$$

Then the eigenvalues of the matrix Q^\dagger are

$$\theta_k = -2 \cos \left(\frac{k\pi}{n+1} \right), \quad k = 1, 2, \dots, n.$$

The proof can be done easily for the matrix A^\dagger similar to Theorem 1. ■

3 The integer powers of the matrices A and A^\dagger

Consider the relations $A = KJK^{-1}$ and $A^\dagger = TJ^\dagger T^{-1}$, where J and J^\dagger are the Jordan's forms of the matrices A and A^\dagger , K and T are transforming matrices of the matrix A and A^\dagger , respectively. Since all the eigenvalues of A and A^\dagger are simple, columns of the transforming matrices K and T are the eigenvectors of the matrices A and A^\dagger , respectively [13]. Also all eigenvalues λ_k and λ_k^\dagger corresponds single Jordan cells $J_i(\lambda_k)$ and $J_i^\dagger(\lambda_k^\dagger)$ in the matrix J and J^\dagger , respectively. Then, we write down the Jordan's forms of the matrices A and A^\dagger

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n) \quad (22)$$

and

$$J^\dagger = \text{diag}(\lambda_1^\dagger, \lambda_2^\dagger, \lambda_3^\dagger, \dots, \lambda_n^\dagger). \quad (23)$$

From (8) and (15), we can write the transforming matrices K and T as

$$K = [x_{kj}] = \begin{cases} T_{k-1} \left(\frac{\delta_j}{2} \right); & k = 1, 2, \dots, n-1 \\ \frac{1}{2} T_{k-1} \left(\frac{\delta_j}{2} \right); & k = n \end{cases} \quad j = 1, 2, \dots, n \quad (24)$$

and

$$T = [y_{kj}] = r_{k-1} U_{k-1} \left(\frac{\psi_j}{2} \right) \quad (25)$$

where $\delta_j = \frac{\lambda_j - a}{b}$, $\psi_j = \frac{\lambda_j^\dagger - a}{b}$ and

$$r_{k-1} = \begin{cases} 1, k-1 \equiv 0 \text{ or } 1 \pmod{4} \\ -1, k-1 \equiv 2 \text{ or } 3 \pmod{4} \end{cases}.$$

Considering (24) and (25), we write down the transforming matrices K and T , respectively,

$$K = \begin{bmatrix} T_0\left(\frac{\delta_1}{2}\right) & T_0\left(\frac{\delta_2}{2}\right) & \cdots & T_0\left(\frac{\delta_{n-1}}{2}\right) & T_0\left(\frac{\delta_n}{2}\right) \\ T_1\left(\frac{\delta_1}{2}\right) & T_1\left(\frac{\delta_2}{2}\right) & \cdots & T_1\left(\frac{\delta_{n-1}}{2}\right) & T_1\left(\frac{\delta_n}{2}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{n-2}\left(\frac{\delta_1}{2}\right) & T_{n-2}\left(\frac{\delta_2}{2}\right) & \cdots & T_{n-2}\left(\frac{\delta_{n-1}}{2}\right) & T_{n-2}\left(\frac{\delta_n}{2}\right) \\ \frac{1}{2}T_{n-1}\left(\frac{\delta_1}{2}\right) & \frac{1}{2}T_{n-1}\left(\frac{\delta_2}{2}\right) & \cdots & \frac{1}{2}T_{n-1}\left(\frac{\delta_{n-1}}{2}\right) & \frac{1}{2}T_{n-1}\left(\frac{\delta_n}{2}\right) \end{bmatrix} \quad (26)$$

and

$$T = \begin{bmatrix} r_0 U_0\left(\frac{\psi_1}{2}\right) & r_0 U_0\left(\frac{\psi_2}{2}\right) & \cdots & r_0 U_0\left(\frac{\psi_n}{2}\right) \\ r_1 U_1\left(\frac{\psi_1}{2}\right) & r_1 U_1\left(\frac{\psi_2}{2}\right) & \cdots & r_1 U_1\left(\frac{\psi_n}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ r_{n-2} U_{n-2}\left(\frac{\psi_1}{2}\right) & r_{n-2} U_{n-2}\left(\frac{\psi_2}{2}\right) & \cdots & r_{n-2} U_{n-2}\left(\frac{\psi_n}{2}\right) \\ r_{n-1} U_{n-1}\left(\frac{\psi_1}{2}\right) & r_{n-1} U_{n-1}\left(\frac{\psi_2}{2}\right) & \cdots & r_{n-1} U_{n-1}\left(\frac{\psi_n}{2}\right) \end{bmatrix}. \quad (27)$$

Denoting j th column of the matrix K^{-1} by τ_j and implementing the necessary transformations, we have

$$\tau_j = \gamma_j \begin{bmatrix} \beta_1 T_{j-1}\left(\frac{\delta_1}{2}\right) \\ \beta_2 T_{j-1}\left(\frac{\delta_2}{2}\right) \\ \beta_3 T_{j-1}\left(\frac{\delta_3}{2}\right) \\ \vdots \\ \beta_{n-1} T_{j-1}\left(\frac{\delta_{n-1}}{2}\right) \\ \beta_n T_{j-1}\left(\frac{\delta_n}{2}\right) \end{bmatrix}, \quad j = \overline{1, n}$$

where $\gamma_j = \begin{cases} 1, j=1 \\ 2, 1 < j \leq n \end{cases}$ and $\beta_k = \frac{1}{2n-2} \begin{cases} 1, k=1, n \\ 2, 1 < k < n \end{cases}$.

Let

$$A^s = K J^s K^{-1} = U(s) = (u_{ij}(s))$$

here

$$s = \begin{cases} s \in \mathbb{N}, & n \text{ odd} \\ s \in \mathbb{Z}, & n \text{ even.} \end{cases}$$

Thus

$$u_{ij}(s) = \gamma_j \sum_{k=1}^n \lambda_k^s \beta_k T_{i-1} \left(\frac{\delta_k}{2} \right) T_{j-1} \left(\frac{\delta_k}{2} \right) \quad (28)$$

where $i = 1, 2, \dots, n-1$; $j = 1, 2, \dots, n$ and

$$u_{ij}(s) = \frac{\gamma_j}{2} \sum_{k=1}^n \lambda_k^s \beta_k T_{i-1} \left(\frac{\delta_k}{2} \right) T_{j-1} \left(\frac{\delta_k}{2} \right) \quad (29)$$

for $i = n$; $j = 1, 2, \dots, n$.

Firstly, we assume that n is positive odd integer ($n = 2p + 1, p \in \mathbb{N}$).

Denoting j th column of the inverse matrix T^{-1} by σ_j and implementing necessary transformations, we have

$$T_j^{-1} = \begin{bmatrix} \mu_1 r_{j-1} U_{j-1} \left(\frac{\psi_1}{2} \right) \\ \mu_2 r_{j-1} U_{j-1} \left(\frac{\psi_2}{2} \right) \\ \vdots \\ \mu_{n-1} r_{j-1} U_{j-1} \left(\frac{\psi_{n-1}}{2} \right) \\ \mu_n r_{j-1} U_{j-1} \left(\frac{\psi_n}{2} \right) \end{bmatrix}$$

where

$$\mu_k = \begin{cases} \frac{1}{2n+2} \psi_{\frac{n+1}{2}+k}^2, & 1 \leq k \leq \frac{n-1}{2} \\ \frac{2}{n+1}, & k = \frac{n+1}{2} \\ \frac{1}{2n+2} \psi_{\frac{3(n+1)}{2}-k}^2, & \frac{n+3}{2} \leq k \leq n \end{cases} \quad k = 1, 2, \dots, n (n = 2p + 1, p \in \mathbb{N}).$$

Let

$$(A^\dagger)^s = T (J^\dagger)^s T^{-1} = W(s) = (w_{ij}(s))$$

here $s \in \mathbb{N}$ ($n = 2p + 1, p \in \mathbb{N}$). Hence

$$w_{ij}(s) = \sum_{k=1}^n \left(\lambda_k^\dagger \right)^s \mu_k r_{i-1} r_{j-1} U_{i-1} \left(\frac{\psi_k}{2} \right) U_{j-1} \left(\frac{\psi_k}{2} \right) \quad (30)$$

where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$.

Secondly we assume that n is positive even integer ($n = 2p, p \in \mathbb{N}$) [8].

Denoting j th column of the inverse matrix T^{-1} by σ_j and implementing necessary transformations, we obtain

$$T_j^{-1} = \begin{bmatrix} \eta_1 r_{j-1} U_{j-1} \left(\frac{\psi_1}{2} \right) \\ \eta_2 r_{j-1} U_{j-1} \left(\frac{\psi_2}{2} \right) \\ \vdots \\ \eta_{n-1} r_{j-1} U_{j-1} \left(\frac{\psi_{n-1}}{2} \right) \\ \eta_n r_{j-1} U_{j-1} \left(\frac{\psi_n}{2} \right) \end{bmatrix}$$

where

$$\eta_k = \frac{4 - \psi_k^2}{2n + 2}, k = 1, 2, \dots, n (n = 2p, p \in \mathbb{N}).$$

Let

$$(A^\dagger)^s = T (J^\dagger)^s T^{-1} = L(s) = (l_{ij}(s))$$

where $s \in \mathbb{Z}$ ($n = 2p, p \in \mathbb{N}$). So

$$l_{ij}(s) = \sum_{k=1}^n \left(\lambda_k^\dagger \right)^s \eta_k r_{i-1} r_{j-1} U_{i-1} \left(\frac{\psi_k}{2} \right) U_{j-1} \left(\frac{\psi_k}{2} \right) \quad (31)$$

where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$.

Corollary 3 *Let*

$$\tilde{A}^\dagger = \begin{bmatrix} 0 & & & b & a \\ & -b & a & b \\ & b & a & -b \\ \ddots & -b & a & b \\ \ddots & a & -b & \\ \ddots & \ddots & \ddots & 0 \end{bmatrix} \quad (32)$$

be the anti-tridiagonal matrices, where $0 \neq b, a \in \mathbb{C}$.

Lemma 4 *Let $0 \neq b, a \in \mathbb{C}$, $n = 2p$, $p \in \mathbb{N}$, A^\dagger and*

$$\tilde{J}^\dagger = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \ddots & & \\ & 1 & & \\ 1 & & & 0 \end{bmatrix}. \quad (33)$$

Then

$$\tilde{A}^\dagger = \tilde{J}^\dagger A^\dagger = A^\dagger \tilde{J}^\dagger. \quad (34)$$

Proof. See [11]. ■

Lemma 5 Let \tilde{A}^\dagger be as in (32). Then

$$\left(\tilde{A}^\dagger\right)^s = \begin{cases} \left(A^\dagger\right)^s, & s \text{ is even} \\ \tilde{J}^\dagger \left(A^\dagger\right)^s, & s \text{ is odd.} \end{cases} \quad (35)$$

Proof. We will use by induction on s . The case $s = 1$ is explicit. Assume that the equality (35) is true for $s > 1$. Now let us show the equality (35) is true for $s + 1$. By the induction hypothesis we possess

$$\left(\tilde{A}^\dagger\right)^{s+1} = \begin{cases} \tilde{A}^\dagger \tilde{J}^\dagger \left(A^\dagger\right)^s, & s + 1 \text{ is even} \\ \tilde{A}^\dagger \left(A^\dagger\right)^s, & s + 1 \text{ is odd.} \end{cases}$$

Since $\tilde{A}^\dagger = A^\dagger \tilde{J}^\dagger$, we get

$$\left(\tilde{A}^\dagger\right)^{s+1} = \begin{cases} \left(A^\dagger\right)^{s+1}, & s + 1 \text{ is even} \\ \tilde{J}^\dagger \left(A^\dagger\right)^{s+1}, & s + 1 \text{ is odd.} \end{cases}$$

■

Theorem 6 Let \tilde{A}^\dagger be n -square complex anti-tridiagonal matrix in (32). If s is odd, then the s -th power of \tilde{A}^\dagger is

$$\chi_{n-i+1,j}^s = \sum_{k=1}^n \left(\lambda_k^\dagger\right)^s \eta_k r_{n-i} r_{j-1} U_{n-i} \left(\frac{\psi_k}{2}\right) U_{j-1} \left(\frac{\psi_k}{2}\right) \quad (36)$$

and if s is even, then the s -th power of \tilde{A}^\dagger is

$$\chi_{i,j}^s = \sum_{k=1}^n \left(\lambda_k^\dagger\right)^s \eta_k r_{i-1} r_{j-1} U_{i-1} \left(\frac{\psi_k}{2}\right) U_{j-1} \left(\frac{\psi_k}{2}\right) \quad (37)$$

for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$.

Proof. Let $\left(\tilde{A}^\dagger\right)^s = \left(\chi_{ij}^s\right)$ and $L = A^\dagger$. We obtain the eigenvalues of A^\dagger as (14) and the entries of the matrix L as (31). Let s be odd interger. If we multiply the equality (31) by \tilde{J}^\dagger from left side, then we possess

$$\left(\tilde{J}^\dagger A^\dagger\right)_{i,k} = \sum_{r=1}^n \left(\tilde{J}^\dagger\right)_{i,r} l_{r,k}(s) = l_{n-i+1,k}(s); \quad k = 1, \dots, n.$$

Therefore we get

$$\begin{aligned} \chi_{n-i+1,j}^s &= l_{n-i+1,k}(s) \\ &= \sum_{k=1}^n \left(\lambda_k^\dagger\right)^s \eta_k r_{n-i} r_{j-1} U_{n-i} \left(\frac{\psi_k}{2}\right) U_{j-1} \left(\frac{\psi_k}{2}\right); \quad i, j = \overline{1, n}. \end{aligned}$$

Let s be even, then the equality (31) is valid by the equality (35). ■

4 Numerical examples

Example 7 If $s = 3$ and $n = 3$. Then

$$A = \begin{bmatrix} a & 2b & 0 \\ b & a & 2b \\ 0 & b & a \end{bmatrix}.$$

We have

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \text{diag}(a + 2b, a, a - 2b)$$

and

$$A^3 = (u_{ij}(s)) = (u_{ij}(3)) = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} = \begin{bmatrix} x & 2y & 4z \\ y & q & 2y \\ z & y & x \end{bmatrix};$$

$$x = a^3 + 6ab^2, \quad y = 3a^2b + 4b^3, \quad z = 3ab^2, \quad q = a^3 + 12ab^2.$$

Example 8 $s = -3$, $n = 4$, $a = 1$ and $b = 2$. Then, we get

$$\begin{aligned} J &= \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \text{diag}(a + 2b, a + b, a - b, a - 2b) \\ &= \text{diag}(5, 3, -1, -3) \end{aligned}$$

and

$$A^{-3} = (u_{ij}(-3)) = \frac{1}{1000} \begin{bmatrix} -326 & 361 & 311 & -676 \\ 180 & -170 & -158 & 311 \\ 156 & -158 & -170 & 361 \\ -169 & 156 & 180 & -326 \end{bmatrix}.$$

Example 9 If $s = 4$, $n = 3$. Then

$$A^\dagger = \begin{bmatrix} a & b & 0 \\ b & a & -b \\ 0 & -b & a \end{bmatrix}.$$

We achieve

$$J^\dagger = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \text{diag}(a - b\sqrt{2}, a, a + b\sqrt{2})$$

and

$$(A^\dagger)^4 = w_{ij}(4) = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} = \begin{bmatrix} x^\dagger & y^\dagger & z^\dagger \\ y^\dagger & q^\dagger & -y^\dagger \\ z^\dagger & -y^\dagger & x^\dagger \end{bmatrix};$$

$$x^\dagger = a^4 + 6a^2b^2 + 2b^4, \quad y^\dagger = 4a^3b + 8ab^3, \quad z^\dagger = -6a^2b^2 - 2b^4, \quad q^\dagger = a^4 + 6a^2b^2 + 2b^4.$$

Example 10 If $n = 4$, $s = 4$, $a = 1$ and $b = 4$, then

$$\begin{aligned} J^\dagger &= \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \text{diag}\left(a - \frac{b}{2}(1 + \sqrt{5}), a - \frac{b}{2}(\sqrt{5} - 1), a - \frac{b}{2}(1 - \sqrt{5}), a + \frac{b}{2}(1 + \sqrt{5})\right) \\ &= \text{diag}(-1 - 2\sqrt{5}, 3 - 2\sqrt{5}, -1 + 2\sqrt{5}, 3 + 2\sqrt{5}). \end{aligned}$$

Therefore

$$(A^\dagger)^4 = l_{ij}(4) = \begin{bmatrix} 609 & 528 & -864 & -256 \\ 528 & 1473 & -784 & -864 \\ -864 & -784 & 1473 & 528 \\ -256 & -864 & 528 & 609 \end{bmatrix}.$$

Example 11 If $n = 4$, $s = -5$, $a = i$ and $b = 1$, then

$$J^\dagger = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \text{diag}\left(i - \frac{1}{2}(1 + \sqrt{5}), i - \frac{1}{2}(\sqrt{5} - 1), i - \frac{1}{2}(1 - \sqrt{5}), i + \frac{1}{2}(1 + \sqrt{5})\right).$$

So

$$(A^\dagger)^{-5} = l_{ij}(-5) = \frac{1}{1000} \begin{bmatrix} 296i & 56 & 192i & 128 \\ 56 & 104i & 72 & 192i \\ 192i & 72 & 104i & 56 \\ 128 & 192i & 56 & 296i \end{bmatrix}.$$

5 Complex Factorizations

The well-known Fibonacci polynomials $F(x) = \{F_n(x)\}_{n=1}^\infty$ are defined in [12] by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad (38)$$

where $F_0(x) = 0$, $F_1(x) = 1$ and $n \geq 3$.

Theorem 12 Let the matrix A be n -square matrix as in (1) with $a := x$ and $b := \mathbf{i}$ where $\mathbf{i} = \sqrt{-1}$. Then

$$\det(A) = (x^2 + 4)F_{n-1}(x) \quad (39)$$

where F_n is n th Fibonacci number.

Proof. Applying Laplace expansion according to the first two and the last two rows of the matrix A , we have

$$\det(A) = x^2 \check{D}_{n-2} + 4x \check{D}_{n-3} + 4 \check{D}_{n-4}$$

here $\check{D}_n = \det(\text{tridiag}_n(\mathbf{i}, x, \mathbf{i}))$. Since

$$\det(\text{tridiag}_n(\mathbf{i}, x, \mathbf{i})) = F_{n+1}(x),$$

we have

$$\begin{aligned}
\det(A) &= x^2 F_{n-1}(x) + 4x F_{n-2}(x) + 4F_{n-3}(x) \\
&= x^2(x F_{n-2}(x) + F_{n-3}(x)) + 4x F_{n-2}(x) + 4F_{n-3}(x) \\
&= (x^2 + 4)(x F_{n-2}(x) + F_{n-3}(x)) = (x^2 + 4)F_{n-1}(x).
\end{aligned}$$

So that, the proof is completed. ■

Corollary 13 *Let the matrix A be as in (1) with $a := x$ and $b := \mathbf{i}$. Then the complex factorization of generalized Fibonacci-Pell numbers is the following form:*

$$F_{n-1}(x) = \frac{1}{x^2 + 4} \prod_{k=1}^n \left(x + 2\mathbf{i} \cos \left(\frac{(k-1)\pi}{n-1} \right) \right). \quad (40)$$

Proof. Since the eigenvalues of the matrix A from (7)

$$\lambda_k = x + 2\mathbf{i} \cos \left(\frac{(k-1)\pi}{n-1} \right), \quad k = 1, 2, \dots, n$$

the determinant of the matrix A can be expressed as

$$\det(A) = \prod_{k=1}^n \left(x + 2\mathbf{i} \cos \left(\frac{(k-1)\pi}{n-1} \right) \right).$$

By considering (40) and Theorem 12, the complex factorization of generalized Fibonacci-Pell numbers is achieved. ■

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